ON SOME GENERAL CONCEPTS OF THE MATHEMATICAL THEORY OF BRITTLE FRACTURE

(O NEKOTORYKH OBSHCHIKH PREDSTAVLENIIAKH MATEMATICHESKOI TEORII KHRUPKOGO RAZRUSHENIIA)

PMM Vol.28, № 4, 1964, pp.630-643

G.I.BARENBLATT
(Moscow)

(Received April 19, 1964)

The theory of brittle fracture originated more than forty years ago in the classical works of Griffith [1 and 2]. Until comparatively recently this theory was looked upon as one more of academic rather than of practical interest, since the number of materials which fail under normal conditions in a brittle manner is relatively small. Recently, however, considerable attention has been devoted to phenomena associated with brittle failure and in particular to the equilibrium and propagation of cracks. It has been discovered that at elevated or low temperatures many structures of commonly used materials which display fully plastic properties in standard tensile tests fail by a quasi-brittle mechanism. The term quasi-brittle fracture means that the failure occurs by means of the propagation of cracks and that the plastic region, although it exists, is very narrow and is concentrated in the immediate neighborhood of the surface crack. In the analysis of a quasi-brittle fracture it is possible to make use of the laws governing a purely brittle fracture by replacing the surface tension by the total density of surface energy expended not only in overcoming intermolecular forces but also in the plastic deformation of the thin surface layer. This fact was first discovered by Irwin [3] and Orowan [4]. Over the last decade a considerable number of papers have been produced on the investigation of the statics and dynamics of elastic bodies with cracks. The results achieved so far in the study of the equilibrium and propagation of cracks enable us to formulate the basic problems of the mathematical theory of brittle fracture in more general form.

The analytical determination of the brittle strength of a given body under a given system of loading must be considered as a problem of the mathematical theory of brittle fracture. First of all we must define the quantitative characteristic of brittle strength. A precise definition of this characteristic is given later. At this stage it must be emphasized that, whatever this definition is, in assessing the brittle strength of the body we must take into account from the very beginning the existence of cracks within the body and determine their effect on the body's strength: brittle fracture takes place by means of the development of defects existing within the body. Therefore, the problem of the equilibrium of elastic bodies with cracks, i.e. the problem of the determining the elastic fields and crack configurations, is of extreme importance in the theory of brittle fracture. The characteristic of these problems is that the surface shape of the cracks in the body under a given load is not specified but is determined, in general, by the complete loading process and by the initial cracks which already exist within the body before loading commences. This factor makes the problems of the theory of cracks essentially nonlinear and extremely difficult: at present effective analytical solutions exist for very few problems.

It should be pointed out that the solution to a problem on the equilibrium of a body with cracks provides much more information than is required in practice: after all, the elastic field and the dimensions of a crack inside the body are of only limited interest. In fact it is only important to know if a body under a given loading has the necessary carrying capacity or not. In mathematical terms we can say that the actual solution to the problem of equilibrium of a body with cracks is of no interest; we only require to know whether or not such solution exists for a given loading. Thus failure implies the onset of conditions which ensure the non-existence of a solution to the appropriate problem of the elastic equilibrium of a body with cracks. These conditions are of an essentially integral nature and are not determined by the local structure of the state of stress anywhere within the body. The approach adopted here to the brittle fracture agrees in general with the global conception of the failure of solid bodies [5].

Some investigators are inclined to see a serious limitation in this theory, in that it does not cover the formation of cracks and the resulting strength criteria depend on the dimensions of the initial cracks within the body. However, this point of view is too simplified: it is based on the incorrect assumption that in all cases, as soon as a crack starts to develop it assumes catastrophic proportions and leads to the complete failure of the structure. In fact the development of cracks in a well designed and manufactured structure is at first stable [6], so that with increase in load the size of the crack at first increases continuously. Under these conditions the strength of the body within certain limits proves to be independent of the initial crack dimensions. The cracks which exist within or on the surface of the structure need not develop catastrophically for the range of working loads specified; if the cracks are sufficiently small and if their stable development is ensured over the loading range, then the brittle strength characteristic of the structure, determined for some more dangerous crack configuration selected on the basic of structural considerations, is independent of the crack dimensions and can be accepted in the design as a predetermined quantity. The theory of brittle failure can be improved by taking into account the development of cracks from micro-defects: such an improvement is of interest in principle. However, in the majority of cases in practice brittle failure occurs as a result of the development of small but nevertheless macroscopic defects [6].

1. Cracks of brittle fracture in solid bodies can be considered as surfaces of discontinuity of the elastic displacement vector. In general, on such a surface all three components of this vector suffer discontinuities. Until recently very detailed studies had been made of cracks of normal discontinuity on the surface of which only the component of the displacement vector normal to the surface of discontinuity suffers discontinuity, and also shear cracks in which discontinuity exists only in the tangential component of the displacement vector.

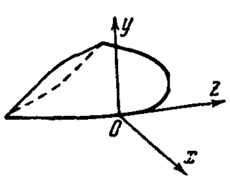

Fig. 1

Consider the neighborhood of an arbitrary point on the contour (*) of a crack in the body (Fig.1). We introduce a natural local coordinate system with origin at the point O : the z-axis is directed along the tangent to the contour of the crack, the y-axis along the normal to the surface of the crack at the point O and the x-axis into the body.

*) By the contour of the crack we mean, as usual, the line bounding the surface of discontinuity of displacements.

It can be shown that the stress distribution at points on the x-axis near the origin is in general of the form

$$\sigma_{yy} = \frac{N}{\sqrt{x}} + O(1), \quad \sigma_{xy} = \frac{T}{\sqrt{x}} + O(1), \qquad \sigma_{yz} = \frac{S}{\sqrt{x}} + O(1)$$
$$_{xx}, \sigma_{xz}, \sigma_{zz} = O(1) \qquad (1.1)$$

where $\sigma_{xx} \ldots \sigma_{zz}$ are components of the stress tensor; N, T and S are the "coefficients of stress intensity" – quantities which depend on the applied loading, the shape of the boundary of the body and of the cracks existing in the body and on the position of the point O, but which are independent of x. It is shown in [7] that the stresses on the contour of the crack must be finite, so that

$$N = T = S = 0 \qquad (1.2)$$

In previous works (see the review [8]), with few exceptions rectilinear cracks of normal discontinuity have been considered, for which $\sigma_{xy} \equiv 0$, $\sigma_{yz} \equiv 0$ on the x-axis, or longitudinal shear cracks, for which $\sigma_{yy} = \sigma_{xy} = \sigma_{xx} = \sigma_{zz} \equiv 0$ everywhere within the body. The condition that $N = 0$ as a fundamental relation which defines the position of the edges of cracks was first proposed in hypotetical form by Khristianovich [9] and proved, on the basis of the principle of virtual displacements, in [10]. This condition signifies the finiteness of stresses and the smooth closure of opposite sides at the tips of a crack of normal discontinuity.

It is natural to divide the surface of the crack into two regions [11] : an inner region where the opposite sides of the crack are far apart and where the cohesive forces are negligibly small, and an end region where the distance between the opposite sides is small and cohesive forces are present. (In the case of quasi-brittle fracture the surface of the crack is taken as the boundary between the plastic region surrounding the crack and the outer elastic region; in this case the part of cohesive forces is played by forces applied by the plastic tip of the crack). By virtue of the linearity of the problem of the theory of elasticity for a specific shape of the body and its cracks, to which the determination of the quantities N, T and S reduces, these quantities can be put in the form

$$N = N_0 + N_a, \qquad T = T_0 + T_a, \qquad S = S_0 + S_a \qquad (1.3)$$

where those with the suffix zero are calculated without taking into account forces of cohesion acting in the end region of the crack and those with the suffix a correspond to the action only of cohesive forces for the same crack configuration.

The cohesive forces increase with increase in the load applied to the body and finally reach some maximum intensity, after which at this point the crack starts to advance. In the study of cracks of normal discontinuity two fundamental hypotheses were made (see the review [8]): the hypothesis of the smallness of the end region and the hypothesis of the "autonomy" (independence of loading under specified conditions) of the shape of the surface of the crack (and consequently, of the distribution of the cohesive forces) in the neighborhood of points at which the intensity of cohesive forces is a maximum. These hypotheses lead to the condition that everywhere on the contour of a crack of normal discontinuity

$$N_0 \leqslant K/\pi$$

where K is the cohesion modulus [11 and 8] – a constant of the material which is an integral characteristic of the cohesive forces for cracks of normal discontinuity and which characterizes resistance of the material to brittle fracture. The state of points on the contour at which $N_0 = K/\pi$ is limiting, so that any change in the load which would have led to an increase in N_0 in fact gives rise to movement of the crack at these points on the contour.

In general the crack may be orientated in any way relative to the applied loads, so that there will be no uniqueness in the shape of the end region at the start of propagation of the crack. In order to determine the conditions for the initiation of the development of a crack we make the following hypothesis which is a natural generalization of the hypothesis of the autonomy of the end region for a crack of normal discontinuity: for any body in which failure occurs by a brittle or quasi-brittle mechanism there exists a universal function of coefficients of intensities of cohesive forces

$$\Phi(-N_a, -T_a, -S_a) \tag{1.4}$$

such that

$$\Phi \leqslant 0 \tag{1.5}$$

at all points on the contours of all cracks within the body. The state of points on the contours of cracks at which $\Phi = 0$, is limiting in the sense that the attainment of this state at some point on the contour makes the cracks move at this point and any change in the load which would have led to the rearrangement of the cohesive forces at that point, which would have made $\Phi > 0$ in fact, by virtue of (1.5) leads to the advance of the crack at that point on the contour. Relations (1.2) and (1.3) enable us to write the limiting condition in the form

$$\Phi(N_0, T_0, S_0) = 0 \tag{1.6}$$

In particular, if the limiting condition corresponds to the constant energy of rupture, so that the density of energy γ expended in forming a new surface crack is constant independently of the manner in which the limiting state is reached at a given point on the contour, then relation (1.6) assumes the form

$$\frac{\pi(1+\nu)}{E}[(1-\nu)(N_0^2+T_0^2)+S_0^2] = \gamma$$

where ν is Poisson's ratio, E is Young's modulus. But the density of surface energy γ is related to the modulus of cohesion K by the relation [11 and 8]

$$K^2 = \pi E\gamma/(1-\nu^2)$$

Therefore, from the preceding relation it follows that

$$\Phi \equiv N_0^2 + T_0^2 + \frac{1}{1-\nu}S_0^2 - \frac{1}{\pi^2}K^2 \tag{1.7}$$

For cracks of normal discontinuity ($T_0 = S_0 = 0$) relations (1.6) and (1.7) lead to the familiar condition [11 and 8]

$$\Phi \equiv N_0 - K/\pi = 0 \tag{1.8}$$

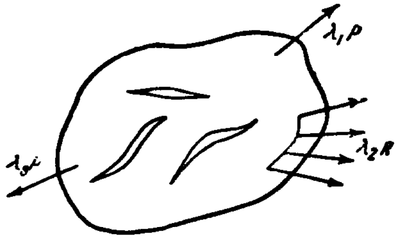

Fig. 2

Relation (1.6) defines the conditions for the commencement of crack development, but, in general, this relation is insufficient to establish the subsequent propagation of the crack.

Suppose that a brittle body possessing a certain initial crack system is subjected to a system of loading s proportional to certain nondimensional parameters $\lambda_1, \lambda_2, \ldots \lambda_s$ (Fig.2) (and also, perhaps, loaded by another system which remains unaltered during the whole loading process now to be considered). The values of the parameters $\lambda_1 = \lambda_2 = \ldots = \lambda_s = 0$ correspond to the initial state of the body. The simplest particular case is when $\lambda_1 = \lambda_2 = \ldots = \lambda_s = \lambda$, the case of so-called proportional loading. Consider the space $\lambda_1 \lambda_2 \ldots \lambda_s$. In this space the curve

$$\lambda_1 = \lambda_1(t), \quad \lambda_2 = \lambda_2(t), \quad \ldots, \quad \lambda_s = \lambda_s(t) \quad (t \text{ is a parameter}) \qquad (1.9)$$

which passes through the origin of coordinates $\lambda_1 = \lambda_2 = \ldots = 0$ at $t = 0$ defines a certain loading path for the body.

The loading path is called active if the quantity ~ for all points on the contours of all cracks in the body does not decrease over the whole loading path.

The problem of the equilibrium of a body with cracks may be formulated as follows: for an initial state of the body which is specified at $t = 0$ and for an initial system of cracks and a given loading path, to determine the elastic field and crack configuration corresponding to some value of $t = t_1 > 0$.

2. Consider the following example. Imagine (Fig.3) an infinite plate with a crack under the action of a uniform stress $\lambda_1 p$ applied at infinity in a direction perpendicular to the crack (λ_1 is a nondimensional loading parameter, p is some constant which has the dimensions of stress). The plate is reinforced by two pairs of wire loops threaded through holes specially drilled in the plate. If there is no initial stress in the loops then the effect of the loops reduces approximately to the action of four pairs of concentrated forces which increase with increase in the loading parameter λ_1 . Here we have the simplest case of a crack of normal discontinuity, for which

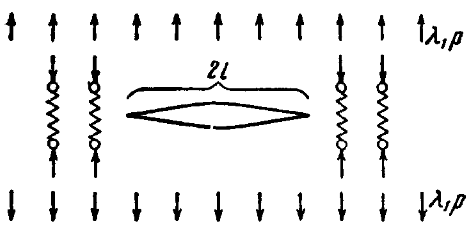

Fig. 3

$$\Phi \equiv N_0 - K / \pi.$$

In the present case of a symmetrical isolated crack [11 and 8] we have

$$N_0 = \frac{\sqrt{2l}}{\pi} \int_0^1 \frac{g(l\xi)\, d\xi}{\sqrt{1 - \xi^2}} \qquad (2.1)$$

where l is the half-length of the crack and $g(x)$ is the distribution of normal stress at the position of the crack in a solid body under the action of the same loading, so that the limiting condition (1.8) assumes the form

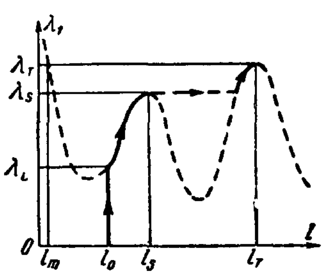

Fig. 4

$$\lambda_1 = \varphi(l) \equiv \frac{K}{\sqrt{2l}}\left[\int_0^1 \frac{g(l\xi)\,d\xi}{\sqrt{1-\xi^2}}\right]^{-1} \tag{2.2}$$

Expression (2.2) difines in general the crack dimension $2l$ corresponding to each value of the parameter λ_1 if the initial length $2l_0$ in the unloaded plate (for $\lambda_1 = 0$) is known, i.e. it gives the solution for the present case to the basic problem of the theory of cracks. Indeed, it can be shown that for this case the curve of the function $\varphi(l)$ is of the form shown schematically in Fig.4. As λ_1 increases, the crack length remains constant and equal to $2l_0$ until the value of the parameter $\lambda_1 = \lambda_L$, is reached, when the limiting conditions at the tips of the crack are reached and relation (2.2) starts to be satisfied. With further increase in the loading parameter λ_1 the crack starts to extend: its length is given by (2.2) and depends continuously on the magnitude of the loading parameter (the crack develops in a stable manner) until the value of $\lambda_1 = \lambda_S$ is reached. It can be seen that with increase in the parameter λ_1 from λ_L to λ_S, in spite of the growth of the crack, no failure of the body occurs: the plate as before remains able to withstand the increasing load.

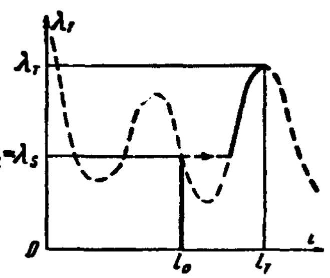

Fig. 5

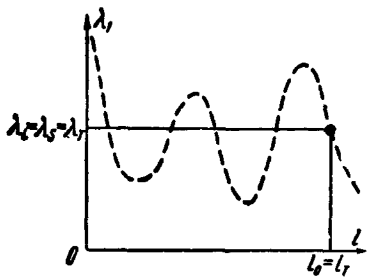

Fig. 6

After the loading parameter reaches the value $\lambda_1 = \lambda_S$ the position alters sonewhat. The solution (in particular the size of the crack) no longer depends continuously on the loading parameter: the crack becomes unstable and increases in a sudden jump. What is important is that even this does not indicate the failure of the body: with further increase in λ_1 the crack continues to grow in a stable manner and the plate remains able to withstand the increasing load until the loading parameter reaches the value $\lambda_1 = \lambda_T$. With any further increase in the loading parameter, no matter how small, the solution to the problem of the equilibrium of the body with a crack no longer exists. Physically, this means that for all $l > l_T$ the limiting state at the tips of the cracks is already reached for values of the loading corresponding to smaller values of the loading parameter λ_1, so that for $\lambda_1 > \lambda_T$ the cohesive forces are no longer able to check the crack development and failure of the body occurs. Thus the nset of failure is associated with the nonexistence of a solution to the problem of the equilibrium of a body with a crack. The limiting value of the loading parameter λ_T defines the strength of a plate with a crack.

For other values of the initial crack length the characteristic values of the loading parameter may coincide. For instance, if l_0 lies on the second un table segment (Fig.5), then $\lambda_L = \lambda_S$, i.e. the development of the crack becomes unstable immediately after the limiting state is reached at the tips of the crack, but failure of the body still does not occur. If l_0 corresponds to the third unstable segment (Fig.6), then $\lambda_L = \lambda_S = \lambda_T$, i.e. the development of the crack immediately becomes unstable and complete failure of the body occurs.

It is very important for the limiting value of the loading parameter λ_T (Fig.4), which is a characteristic of the strength of the body, to be the same for all values of the initial crack size within the range $l_m < l_0 < l_T$. If $l_0 < l_m$ or $l_0 > l_T$, then immediately after the limiting state is reached at the tips of the crack a catastrophic crack growth occurs leading to total failure of the body. The corresponding value of the loading parameter λ_F for $l_0 < l_m$ is greater than λ_T, and for $l_0 > l_T$ it is less than λ_T. Therefore for the structure under consideration we can formulate the following general conclusion.

1. In the unloaded plate the crack length must be less than $2l_T$.

2. If condition 1 is satisfied, then the application of any load less than $\lambda_T p$ will not cause failure.

Thus for the present simple case the practical recommendations are as follows: plates with a crack length greater than $2l_T$ should be rejected (*); as a strength limit the value of $\lambda_T p$ may be taken; this value is independent of the crack length. For $l_0 < l_m$ such a definition of strength limit is on the safe side.

One often encounters assertion that the condition $N_0 = K/\pi$ always leads to the start of catastrophic crack development and failure of the body.

The example just considered shows that the start of local failure, i.e. the development of cracks within the body, does not in general coincide with the start of unstable crack development, and the start of unstable catastrophic growth of the cracks existing within the body does not in general coincide with total failure of the body. In other words the failure of a body is determined not by the local structure of the state of stress at any point of the body, but by the essentially integral condition of the nonexistence of a solution to the problem of the equilibrium of a body containing cracks.

The acceptance of the condition of local failure as a criterion of failure of the body can lead to a considerable underestimate of the body's carrying capacity. The example also shows that the theory of brittle fracture makes it possible to derive a universal characteristic of the strength of a structure which can be used in the calculations and which is independent of the size of the initial crack.

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*) This requirement is obvious: it means simply that the crack should not extend beyond the reinforcing loops.

3. Let us return to the general case. For a given body with a given crack configuration and a given system of loading let us consider a surface of local failure

$$F_L(\lambda_1, \lambda_2, \ldots, \lambda_s) = 0 \tag{3.1}$$

in the space $\lambda_1 \lambda_2 \ldots \lambda_s$ which has the following properties: for any loading represented by points inside this surface and reached by a loading path which also lies wholly within this surface, no development of a crack system takes place at any point on the contour and no crack growth takes place within the body. In transferring to any point in the region outside the surface (3.1) one or other of these developments of a system of cracks takes place. In particular, if there are no stresses in the body for $\lambda_1 = \lambda_2 = \ldots = 0$, then the origin of coordinates must be inside the region bounded by the surface of local failure. In the example considered in Section 1 the equation of the surface of local failure is of the form

$$\lambda_1 - \lambda_L = 0 \tag{3.2}$$

In general case the surface of local failure can be constructed as follows.

For a given configuration of the crack system within the body the quantities N_0, T_0 and S_0 are linear functions of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_s$:

$$N_0 = \sum_{i=1}^{s} \lambda_i n_i + N_{00}, \quad T_0 = \sum_{i=1}^{s} \lambda_i t_i + T_{00}, \quad S_0 = \sum_{i=1}^{s} \lambda_i s_i + S_{00} \tag{3.3}$$

where n_i, t_i and s_i are the magnitudes of N_0, T_0 and S_0 corresponding to unit values of the parameter λ_i and zero values of the remaining parameters; N_{00}, T_{00} and S_{00} are the magnitudes of N_0, T_0 and S_0 corresponding to zero values of all s parameters λ_i.

If we substitute expressions (3.3) into the expression for the function $\Phi(N_0, T_0, S_0)$, we obtain some function of the parameters $\lambda_1, \ldots, \lambda_s$ and of the coordinates of the points on the contours at which the quantities n_i, t_i and s_i are evaluated. The vanishing of $\Phi(N_0, T_0, S_0)$ determines the onset of a limiting state at the given point on the contour of a crack, so that the relation $\Phi = 0$ determines a one-parameter family of surfaces in the space $\lambda_1 \lambda_2 \ldots \lambda_s$. The surface bounding the intersection of the inner regions of the whole family is the surface of local failure. It defines the domain of loads for which no development of the crack system in the body takes place.

On transfer to the region outside the surface of local failure, growth of crack system takes place at one point at least within the body. Let us first examine a more specific, but still sufficiently wide class of problems for which the direction of propagation of the cracks is pre-determined (for example, problems in which the symmetry of the body and of the loading ensures the development of plane cracks, or problems of cracks in jointed bodies). In such problems for the case of an active loading path the elastic field and the configuration of the crack system within the body is independent of the loading path and depends only on the final loading. Therefore, in addition to the surface of local failure it is expedient to introduce a surface of loss of stability and a suface of total failure defined as follows. The surface of loss of stability in the space $\lambda_1\lambda_2\ldots\lambda_s$ is defined by

$$F_S(\lambda_1, \lambda_2, \ldots, \lambda_s) = 0 \tag{3.4}$$

This surface has the property that on travelling along any loading path lying inside this surface the cracks develop in a stable manner, so that the configuration of the crack system varies continuously with travel along the loading path. When the loading path leaves the region inside this surface at least one of the cracks becomes unstable. In the example considered in Section 2 the equation of the surface of the loss of stability is

$$\lambda_1 - \lambda_S = 0 \tag{3.5}$$

The surface of loss of stability defines the range of values of loading parameters within which no catastrophic growth of the cracks existing in the body takes place.

The surface of total failure in the space $\lambda_1\lambda_2\ldots\lambda_s$ is defined as the surface

$$F_T(\lambda_1, \lambda_2, \ldots, \lambda_s) = 0 \tag{3.6}$$

such that a solution exists to the problem of the equilibrium of a body possessing cracks for all points inside this surface approaching along any loading path which remains within the region contained inside this surface and no solution exists for points outside this surface.

In the example of Section 2 the equation of the surface of total failure is

$$\lambda_1 - \lambda_T = 0 \tag{3.7}$$

The surface of total failure is the most important characteristic of the strength of a body for a given type of loading – it defines the safe range of values of the loading parameters for which failure of the body does not occur. It is of practical interest to determine the surface of total failure, since the start of crack growth within the body or even the loss of stability of the crack system, as has already been indicated, is immaterial; what is essential is to establish when the carrying capacity of the body will be exceeded, i.e. when the body will fail.

Thus, for problems in which the direction of crack propagation is predetermined, the determination of the basic strength characteristic – the surface of total failure – reduces to the determination of the region of existence of a solution to the problem, formulated precisely in a mathematical sence, of the equilibrium of a body possessing cracks.

The problem becomes very much more complicated in the general case of a curvilinear crack surface, when the direction of crack propagation is not predetermined and when the shape of the surface of the cracks depends on the complete loading path and not only on the position of its end point. In this case a further hypothesis becomes essential which determines the direction of propagation of the cracks; one such hypothesis will be discussed below. It is not essential to know exactly what the hypothesis is, and for our present purposes it is sufficient to assume that there exists some condition which determines the direction of propagation of a crack in an infinitely small region on its contour for the configuration of crack systems and loading existing at any given moment.

Let us take some loading path $\lambda_1 = \lambda_1(t)$, $\lambda_2 = \lambda_2(t), \ldots, \lambda_s = \lambda_s(t)$ and move along it from the origin of coordinates $t = 0$. For sufficiently small values of t the configuration of the crack system within the body does not alter, since at no point on the contours of the cracks is the limiting state reached. Suppose that at $t = t_*$ the limiting state is reached at one point at least. Consider a neighboring point on the active loading path under discussion, corresponding to $t = t_* + \Delta t$. Due to the assumed condition the direction of crack propagation is known at all points where the limiting state is reached. Therefore in general, for a sufficiently small Δt, we can find a shape of crack contour within the body so infinitely close to the initial shape that the limiting condition (1.6) will be satisfied at all new points on the crack contours. Now we apply a new increment $\Delta_1 t$ and repeat the preceding operation. It might happen at some stage (in particular, the first) that no matter how small the increment Δt there does not exist a similar shape of the contours such that the limiting condition (1.6) is satisfied at all new points on the crack contours. This indicates either the loss of stability of one of the cracks (i.e. the nonexistence of an infinitely close solution), or failure (i.e. the nonexistence of a solution at all). In general it is not possible to distinguish these two forms of nonexistence of a solution within the framework of the present purely statical theory, since essentially the process of transition of a crack from one unstable state to another stable state is a dynamic problem. If the direction of crack growth is predetermined, the actual dynamic process of crack development is not essential for the analysis, and we can confine our attention to information concerning the initial and final states which must satisfy the statical conditions. In the general case the shape of the surface of a crack at the end of a dynamic process of crack development is determined by the whole course of the dynamic process.

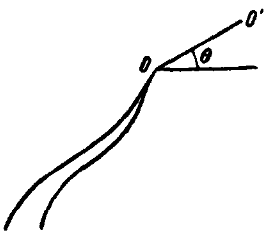

Fig. 7

Thus, generally, it is possible in principle only to specify the limiting value of the parameter t for a given loading path (if such a value exists) at which the solution to the problem of the equilibrium of an elastic body possessing cracks first ceases to depend continuously on the parameter t. Whether this limiting value corresponds to total failure or only a catastrophic unstable growth of one of the cracks without the body losing its carrying capacity, is not possible in general to establish.

An analysis of experimental data enables us to formulate a condition which determines the direction of propagation of curvilinear cracks.

If the problem of the equilibrium of a body containing cracks is plane, so that the surface of a crack is cylindrical, then as a basic hypothesis determining the direction of propagation of the crack we can take the hypothesis of local symmetry [12] of the state of stress in the neighborhood of the new tip of the crack. This hypothesis comprises the following assumption. Suppose there exists a curvilinear crack (Fig.7) at the tip O of which the limiting state has been reached. Then the angle θ at which the crack will spread with an infinitely small increase in load is determined by the condition of symmetry of the state of stress in a small region near the new tip O' of the crack about the line OO'. In particular, this implies the absence of transverse shear in this region, i.e. the vanishing of shear stresses in the xy-plane close to the point O' on the line of propagation of the crack. In particular, on the basis of this condition, in the absence also of longitudinal shear, the crack propagates in the direction of the maximum tensile stress, and in the absence of normal discontinuity (a longitudinal shear crack) the crack propagates in the direction of the maximum shear stress. The local symmetry hypothesis has been used in the solution of specific problems in [13 and 14]. The position of the point O' is determined by applying the limiting condition (1.6) at this point, which, by virtue of the fact that the coefficient of intensity of transverse shear stresses T_0 at the point O' is zero, may be written in the form

$$\Phi\,(N_0', 0, S_0')=0 \tag{3.8}$$

where N_0' and S_0' are the values of the coefficients of stress intensity at the point O'. If in particular there is no longitudinal shear, condition (3.8) assumes the usual form

$$N_0' = K\,/\,\pi \tag{3.9}$$

Conversely, in the absence of normal discontinuity, this condition becomes

$$S_0' = L/\pi \tag{3.10}$$

where L is an appropriate constant of the cohesion modulus type for cracks of longitudinal shear. If we start from the condition of constant density of surface energy, then, by virtue of (1.7)

$$L = K\sqrt{1-\nu} \tag{3.11}$$

As an illustrative example let us construct the surfaces of local failure, loss of stability and total failure for an infinite plate containing one isolated crack in tension under a uniform stress $\lambda_1 p$ applied at infinity in a direction perpendicular to the line of the crack and with concentrated forces $\lambda_2 P$ applied at opposite points on the surface of the crack at its center. The length of the crack in the unloaded plate is $2l_0$.

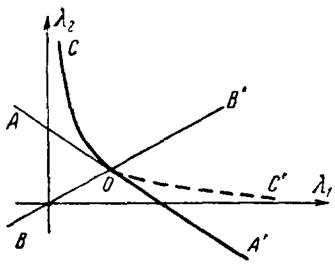

Fig. 8

A simple analysis based on the method used by N.I. Muskhelishvili gives

$$N_0 = \frac{\lambda_1 p\sqrt{l}}{\sqrt{2}} + \frac{\lambda_2 P}{\sqrt{2}\pi\sqrt{l}} \tag{3.12}$$

On the basis of condition (1.8) the surface (in the present case, of course, a line) of local failure (the line AA' in Fig.8) is given by

$$\frac{\lambda_1 p\sqrt{l_0}}{\sqrt{2}} + \frac{\lambda_2 P}{\sqrt{2}\pi\sqrt{l_0}} = \frac{K}{\pi} \tag{3.13}$$

After the limiting state is reached at the tips of the crack the half-length l is given by

$$\frac{\lambda_1 p\sqrt{l}}{\sqrt{2}} + \frac{\lambda_2 P}{\sqrt{2}\pi\sqrt{l}} = \frac{K}{\pi} \tag{3.14}$$

Differentiating Expression (3.14) with respect to λ_1 and λ_2, we obtain

$$\frac{\partial l}{\partial \lambda_1}\left(\lambda_1 p - \frac{\lambda_2 P}{\pi l}\right) + 2pl = 0, \quad \frac{\partial l}{\partial \lambda_2}\left(\lambda_1 p - \frac{\lambda_2 P}{\pi l}\right) + \frac{2P}{\pi} = 0 \qquad (3.15)$$

so that when

$$\lambda_1 p - \frac{\lambda_2 P}{\pi l} < 0 \qquad (3.16)$$

the derivatives $\partial l/\partial\lambda_1$ and $\partial l/\partial\lambda_2$ are simultaneously positive. Since $l \geqslant l_0$ always, it follows that at all points on the line of local failure lying below the point O, its point of intersection with the straight line

$$\lambda_1 p - \frac{\lambda_2 P}{\pi l_0} = 0 \qquad (3.17)$$

(the line BB' in Fig.8), the crack, having reached a moving-equilibrium state, immediately becomes unstable. The coordinates of this intersection point O are

$$\lambda_1 = \frac{K}{p\pi\sqrt{2l_0}}, \qquad \lambda_2 = \frac{K\sqrt{l_0}}{P\sqrt{2}} \qquad (3.18)$$

It can readily be seen that the line

$$\lambda_1 p - \frac{\lambda_2 P}{\pi l} = 0 \qquad (3.19)$$

where l is given by Equation (3.14), is a hyperbola

$$\lambda_1\lambda_2 = K^2 / 2\pi p P \qquad (3.20)$$

passing through the point O (the line CC' in Fig.8), so that the line of loss of stability below the point O coincides with the line of local failure (3.13). Above the point given by (3.18) this line coincides with the hyperbola (3.20). Also, solving Equation (3.14) for $\sqrt{l}$, we find that

$$\sqrt{l} = \frac{\sqrt{2}}{\lambda_1 p}\left\{\frac{K}{2\pi} \pm \left(\frac{Pp}{2\pi}\left[\frac{K^2}{4\pi^2} - \frac{\lambda_1\lambda_2 Pp}{2\pi}\right]\right)^{1/2}\right\} \qquad (3.21)$$

so that above the hyperbola (3.20) the solution (3.21) becomes complex and a real solution ceases to exist. It follows that the line of total failure as far as the point O coincides with the hyperbola (3.20). On crossing the line of loss of stability, which below the point O coincides with the line of local failure (3.13), the derivatives $\partial l/\partial\lambda_1$ and $\partial l/\partial\lambda_2$ become negative and it follows, therefore, that values of λ_1 and λ_2 lying between the line of local failure (3.13) and the hyperbola (3.20) correspond to values of $l < l_0$, which is impossible. This means that in this case the line of total failure coincides everywhere with the line of loss of stability.

Note that the hyperbola (3.20) is the envelope of the lines of local failure (3.13) for various values of the parameter l_0, the initial half-length of the crack. Therefore, if the loading path is such that the crack, on reaching a moving-equilibrium state, develops in a stable manner, then the line of total failure which in this case coincides with the hyperbola (3.20) is independent of the initial crack length and in this respect is universal.

4. We shall now consider the problem of similarity in brittle and quasi-brittle fracture. This aspect of the subject is of considerable importance

since, as a result of the very real mathematical difficulties inherent in problems of brittle fracture it is not possible to count on being able to construct all the possible surfaces of failure by analytical methods for rather complex structures. It seems that in the majority of cases these and other characteristics of brittle strength must be determined from experiments with models. For this purpose a model of the structure in question is prepared, cracks are made in the model and the model is loaded along various loading paths. By observing the behavior of the cracks we can establish one point on the surface of local failure by recording the load at the instant when one crack starts to move. If one of the cracks first starts to develop in an unstable manner, the point obtained lies on the surface of loss of stability. By taking the model to failure by various loading paths we can obtain a number of points on the surface of total failure. In practice we would confine our investigations to the more dangerous loading paths likely to be encountered by the structure.

The nondimensional loading parameters λ_i always occur in a product with dimensional multipliers which characterize the loading applied to the body. It usually happens that these quantities have the dimensions of stress, force per unit length (tension) or simply force. In order to be specific, suppose that λ_1 is multiplied by p which has the dimensions of stress $[p] = FL^{-2}$, λ_2 by R which has the dimensions of force per unit length $[R] = FL^{-1}$ and λ_3 by P which has the dimensions of force $[P] = F$. Then the equation of any of the surfaces of failure may be written in the form

$$F(\lambda_1 p, \lambda_2 R, \lambda_3 P, \ldots, K, d) = 0 \qquad (4.1)$$

where K is the modulus of cohesion and d is a characteristic dimension of the body.

Note that apart from K and d there may be other quantities in the arguments of (4.1); K and d, however, are essential. By virtue of the Π-theorem in dimensional analysis [15], relation (4.1) may be written in the form

$$\Theta(\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_i, \Pi_j, \Pi_k \ldots) = 0 \qquad (4.2)$$

$$\Pi_1 = \frac{\lambda_1 d^{1/2} p}{K}, \quad \Pi_2 = \frac{\lambda_2 R}{K d^{1/2}}, \quad \Pi_3 = \frac{\lambda_3 P}{K d^{3/2}}, \ldots \quad \Pi_i = \frac{\lambda_i}{\lambda_1}, \quad \Pi_j = \frac{\lambda_j}{\lambda_2}, \quad \Pi_k = \frac{\lambda_k}{\lambda_3}$$

where λ_i, λ_j and λ_k, respectively, are other parameters which are multiplied by quantities having the dimensions of stress, force per unit length and force (it is always possible to arrange for these dimensional multipliers to be equal respectively to p, R and P). We shall now consider two bodies, determine their brittle strength and indicate the conditions of similarity. We shall refer to one of the bodies as the actual structure and the other as the model; corresonding quantities will be denoted by the suffices n and m. Firstly, the condition of geometrical similarity of shape of both bodies and of the initial cracks must be satisfied. In addition, the nondimensional parameters of similarity for the model and the actual structure must be equal, i.e. $(\Pi_\alpha)_m = (\Pi_\alpha)_n$.

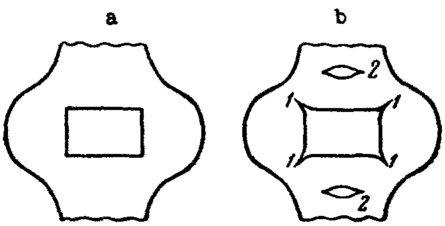

Fig. 9

This relation enables us to translate the values of the strength characteristics for the model determined under conditions of brittle and quasi-brittle fracture to the corresponding quantities for the actual structure.

Thus the correct procedure for analyzing a structure for brittle failure is as follows. On the basis of an analysis of the geometrical shape of the structure and the conditions of its manufacture a representation of the structure is decided upon, which repreoduces the defects typical of its manufacturing process. It must be emphasized that the shape of the structure as shown in Fig.9a should be considered with reproduction of surface cracks 1, resulting from machining, as well as internal cracks 2, resulting from welding and the actual casting process etc. (Fig.9b). Then, theoretically or by model testing, the surfaces of failure for the chosen representation, or at least one or more essential points on this surface, are determined. Then, if the maximum load to be applied to the structure is given, restrictions are placed on the sizes of defects so that the given loading lies inside the surface of total failure or, if a larger safety margin is required, inside the surface of loss of stability. Conversely, if a particular structure and its manufacturing process are specified, the maximum load which ensures no failure of the structure, or even crack stability, is determined.

The preceding discussion appears to explain fully the state of affairs of the purely statical aspect of the mathematical theory of brittle fracture. Problems in the statical theory may be formulated in a precise manner mathematically or easily reproduced experimentally. It would seem that one important problem of the mathematical theory of brittle fracture should be to develop methods for finding upper or lower bounds for the brittle strength characteristics. The problem becomes much more complex when the kinetics of fracture and crack propagation are considered. The point is that in many problems on the kinetics of fracture, apart from increased complications in the mathematical aspects of the problem, the plastic region surrounding the crack becomes significant. In spite of the fact that this region is narrow, the process which takes place whithin it can no longer be simply represented by means of finite relations defining the limiting conditions, as was the case in the static theory. The behavior of the plastic layer surrounding the crack is described in its turn by differential equations which must conform with the differential equations of elastic deformation outside the plastic zone. The difficulty of these problems is a consequence of the incomplete development in the theory of plasticity under conditions of complex states of stress (and particularly the state of stress which exists near the tip of the crack). The narowness of the plastic region provides some hope of the possibility of being able to make use of boundary layer methods.

It is important to assess the effect of viscosity and temporary effects in general both in the mass of the material and in particular in the surface zones of cracks. In the investigation of this effect the solution to the problem depends very much on the so-called sustained strength. The difficulties encountered here are completely analogous to those which arise in the study of kinetic fracture; very promising studies of this problem have been started by Kachanov [16 and 17].

In conclusion the author wishes to record his special gratitude to S.S. Grigorian whose valuable advice made no small contribution to the solution of the problems studied in this paper. The author expresses his thanks also to B.M. Malyshev, R.L. Salganik and V.A. Gorodtsov for their helpful discussions.

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Translated by J.K.L.